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On 3-Closure of 3′-Homogeneous Finite Groups

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A finite group G is said to be 3′-homogeneous if $[N_G(H)]/[C_G(H)]$ is a 3′ group for every 3′ subgroup H of G. A group G is 3-closed if the subset consisting of 3-elements is a subgroup of G.

Throughout this paper, let G denote a 3′-homogeneous finite group such that $3 \mid |G|$.

Since $PSL(2, 2^{n+1})$, $n \geqslant 1$, is a 3′ homogeneous group which is not 3-closed; it is not true that 3′-homogeneous groups are 3-closed. Thus some extra conditions are necessary to guarantee that G is 3-closed.

Let G_3 denote a Sylow 3-subgroup of G. If G is 3-closed, then clearly $C_G(x) \subseteq N_G(G_3)$ for all $x \in G_3^{\#}$. Theorem A below shows this is also a sufficient condition for G to be 3-closed if G_3 is not cyclic. Again $G = PSL(2, 2^{n+1})$, $n \geqslant 1$, shows that G_3 noncyclic is necessary.

THEOREM A. *Let G be a finite 3′ homogeneous group. If G_3 is a Sylow 3 subgroup of G, assume G_3 is noncyclic. Then G is 3-closed, if $C_G(x) \subseteq N_G(G_3)$ for all $x \in G_3^{\#}$.*

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Throughout this section assume G is a minimal counterexample to Theorem A. We will argue by contradiction to show G does not exist.

If W is a subgroup of G, and j is a prime; let W_j denote a Sylow j subgroup of W.

Baer [1] has shown the following: (1) subgroups and factor groups of 3′ homogeneous groups are 3′ homogeneous, (2) if K is a normal subgroup of a group H such that K is 3-closed and H/K is 3-closed, then H is 3-closed.

The minimality of $|G|$ implies if N is a proper subgroup of G satisfying the hypothesis of Theorem A, then we may assume N is 3-closed.

Let H be any proper subgroup of G such that $3 \mid |H|$. Choose notation so that $H_3 \subseteq G_3$. Thus $G_3 \cap H = H_3$. Suppose $y \in C_H(x)$ where $x \in X_3^{\#}$. Then $y \in N_G(G_3)$. Thus $H_3^y = (G_3 \cap H)^y = G_3^y \cap H^y = G_3 \cap H = H_3$. Hence H is a $3'$-homogeneous group such that $C_{H_3}(x) \subseteq N_H(H_3)$ for any $x \in H_3^{\#}$.

Lemma 1. *Let G be a minimal counterexample of Theorem* A; *then* $O_{3'}(G) = 1$.

Proof. Suppose $O_{3'}(G) > 1$, then $G_3 \subseteq N_G(O_{3'}(G))$. However $[N_G(O_{3'}(G))]/[C_G(O_{3'}(G))]$ is a $3'$ group. Hence, $G_3 \subseteq C_G(O_{3'}(G))$. Suppose $yO_{3'}(G) \in [C_G(xO_{3'}(G))]/O_{3'}(G)$ where $x \in G_3^{\#}$. Then $x^y = xz$ where $z \in O_{3'}(G)$. Since $z \in C_G(x)$, $(x^{|\langle z \rangle|})^y = x^{|\langle z \rangle|}$. However, $(|\langle z \rangle|, 3) = 1$ implies $x^y = x$. Hence $y \in C_G(x)$ which implies $y \in N_G(G_3)$. Thus, $yO_{3'}(G) \in [N_G/O_{3'}(G)][G_3O_{3'}(G)]/O_{3'}(G)]$. Since $G_3 \simeq [G_3O_{3'}(G)]/O_{3'}(G)$, we see $G/O_{3'}(G)$ satisfies the hypothesis of Theorem A. The minimality of $|G|$ implies $G/O_{3'}(G)$ is 3-closed. However $O_{3'}(G)$ is trivially 3 closed. Hence G is 3-closed which contradicts the choice of G.

Lemma 2. *Assume G is a minimal counterexample to Theorem* A. *Let V be a maximal proper normal subgroup of G. Then V is 3-closed and $G/V \simeq PSL(2, 2^{2n+1})$, $n \geqslant 1$. If $V_2 > 1$, then $G = C_G(V_2)V$.*

Proof. G/V is a simple $3'$ homogeneous group. If $(|G/V|, 3) = 1$, then $V_3 = G_3$, so the minimality of $|G|$ implies V is 3-closed. However G/V is trivially 3-closed. Hence G is 3-closed. Thus, $3 \mid |G/V|$. Theorem A [2] implies $G/V \simeq PSL(2, 2^{2n+1})$, $n \geqslant 1$, or $G/V \simeq Z_3$. Since a Sylow 3 subgroup of $PSL(2, 2^{2n+1})$, $n \geqslant 1$, is cyclic, we see $3 \mid |V|$.

We first show V is 3-closed. If $V = V_3$, the result is immediate. Hence, assume $V \neq V_3$. Suppose $V_j > 1$, where j is any prime $j \neq 3$. Since $V \triangle G$, Theorem 1.3.7 [3] implies $G = N_G(V_j)V$. Hence $|G_3| = |N_G(V_j)|_3 \, |V|_3 / |N_V(V_j)|_3$(2.1).

Let $H = (N_G(V_j))_3$ and choose notation so that $H \subseteq G_3$. Since $V \triangle G$, Theorem 1.3.8 [3] implies $G_3 \cap N = V_3$. Thus $G_3 \supseteq HV_3$. Since $|V_3 \cap H| \leqslant |N_V(V_j)|_3$, Eq. (2.1) implies $G_3 = HV_3$. Now G $3'$-homogeneous implies $H \subseteq C_G(V_j)$. Hence $V_j \subseteq N_G(G_3)$. However $V \triangle G$ implies V_j normalizes $G_3 \cap V = V_3$. Thus $|V_j| \mid |N_V(V_3)|$ (2.2).

Since j was an arbitrary prime, $j \neq 3$, such that $V_j \neq 1$, and since all the Sylow 3 subgroups of V are conjugate, (2.2) implies $|V|_{3'} \mid |N_V(V_3)|$. Hence V is 3-closed.

If $|G/V| = 3$, then G/V and V 3-closed imply G is 3-closed. Hence $G/V \simeq PSL(2, 2^{2n+1})$, $n \geqslant 1$.

Suppose $V_2 \neq 1$. As in the derivation of Eq. (2.1); $G_3 = HV_3$ where $H = (N_G(V_2))_3$. Again G $3'$ homogeneous implies $H \subseteq C_G(V_2)$. Now

$G = N_G(V_2)V$ implies $C_G(V_2)V \triangle G$. Since $| C_G(V_2)V |_3 = | G_3 |$ and G/V is simply we see $G = C_G(V_2)V$.

Proof of Theorem A. We use all the notation of Lemma 2. For any set $A \subseteq G$, let $\bar{A}$ denote the image of A in G/V.

We claim it is sufficient to show there is a 2 group F such that $|F| \geqslant 4$, $F \cap V = 1$, and $F \subseteq C_G(V_3')$ where V_3' is a nonidentity subgroup of V_3. Indeed, suppose F exists. Then $F \subseteq N_G(G_3)$. Since $F \cap V = 1$, $|\bar{F}| \simeq |F|$. Hence $|\bar{F}| \geqslant 4$. Further $\bar{F} \subseteq N_{\bar{G}}(\bar{G}_3)$ implies $4 \mid | N_{\bar{G}}(\bar{G}_3)|$. However, Lemma 2 implies $\bar{G} \simeq PSL(2, 2^{2n+1})$, $n \geqslant 1$. Hence $| N_{\bar{G}}(\bar{G}_3)|_2 = 2$. Thus G would not exist, so Theorem A would be proved.

The rest of the proof is devoted to showing that F exists.

Suppose $V_2 = 1$. Then $G_2 \simeq \bar{G}_2$ and $\bar{G}_2$ is isomorphic to a Sylow 2 subgroup of $PSL(2, 2^{2n+1})$, $n \geqslant 1$. Hence G_2 is elementary abelian of rank at least 3. Since $G_2 \subseteq N_G(V_3)$, G_2 normalizes $\Omega_1(Z(V_3))$. Since G_2 is elementary abelian, of rank at least 3, applications of Theorem 5.3.16 [3] imply there is a subgroup F of G_2 such that $| G_2/F | \leqslant 2$ and $C_F(\Omega_1(Z(V_3)) = V_3' > 1$. Now $| G_2 | \geqslant 8$ implies $|F| \geqslant 4$. Since $G_2 \cap V = 1$, $F \cap V = 1$ and the theorem is proved in this case.

Hence we may assume $V_2 \neq 1$. Lemma 2 yields $G = C_G(V_2)V$. As in Lemma 2, we may choose notation so that $G_3 = HV_3$ where $H = N_{G_3}(V_2) \subseteq C_G(V_2)$. Lemma 2 also implies G_3/V_3 is cyclic. Thus $G_3/V_3 \simeq H/H \cap V_3$ implies there is an element $g \in H$ such that $G_3 = \langle g \rangle V_3$.

Let $R = C_G(V_2)$, and let $K = C_V(V_2)$. Since $G = RV$; $R/K \simeq G/V \simeq PSL(2, 2^{2n+1})$, $n \geqslant 1$. If A is a subset of R, let $\bar{\bar{A}}$ denote the image of A in R/K.

If R_2 is a Sylow 2 subgroup of R, then $R_2 \cap K = Z(V_2)$. Since $\bar{\bar{R}} \simeq PSL(2, 2^{2n+1})$, $n \geqslant 1$; $\bar{\bar{R}}_2$ is elementary abelian of rank at least 3. Further, there is a cyclic subgroup $\langle \bar{\bar{\gamma}}_1 \rangle$ such that $|\langle \bar{\bar{\gamma}}_1 \rangle| = | \bar{\bar{R}}_2 | - 1$ and $\langle \bar{\bar{\gamma}}_1 \rangle$ acts transitively on $\bar{\bar{R}}_2^{\#}$. Let γ_1 be a preimage of $\bar{\bar{\gamma}}_1$. Then $R_2^{\gamma_1} \subseteq R_2K$. Hence γ_1 normalizes R_2K.

Since R_2 is a Sylow 2 subgroup of R_2K, $R_2^{\gamma_1} = R_2^k$ for some $k \in K$. Let $\gamma_2 = \gamma_1 k^{-1}$. Then $\langle \gamma_2 \rangle \subseteq N_R(R_2)$ and $\langle \bar{\bar{\gamma}}_2 \rangle = \langle \bar{\bar{\gamma}}_1 \rangle$. Let $\langle \gamma \rangle = O_{2'}(\langle \gamma_2 \rangle)$. Then $\langle \gamma \rangle \subseteq N_R(R_2)$ and $\langle \bar{\bar{\gamma}} \rangle = \langle \bar{\bar{\gamma}}_1 \rangle$. Hence $\langle \gamma \rangle$ is a subgroup of odd order which acts transitively on the elements of $R_2^{\#}/Z(V_2)$.

Now R_2 acts on V_3. Let τ be an involution in $R_2 \cap K = Z(V_2)$. We claim $C_{V_3}(\tau) = 1$. Suppose $C_{V_3}(\tau) = V_3'$ where $V_3' \neq 1$. Then $C_G(\tau) \supseteq \langle g, V_3' \rangle$. Let T be a Sylow 3 subgroup of $C_G(\tau)$, where $T \supseteq \langle g, V_3' \rangle$. Lemma 1 implies $| C_G(\tau)| < | G |$. Thus if T is noncyclic, then $T \triangle C_G(\tau)$. Hence, $R_2 \subseteq N_G(T)$ so that $\bar{R}_2 \subseteq N_G(\bar{T})$. However, $G_3 = \langle g \rangle V_3$ implies $\bar{T} = \bar{G}_3$. Thus, $| \bar{R}_2 | \mid | N_G(\bar{G}_3)|$. Since $| \bar{R}_2 | \geqslant 8$, this is a contradiction. Hence T is cyclic. Now $V_3 \triangle G$ implies $V_3' = C_G(\tau) \cap V_3$ is a normal cyclic subgroup of $C_G(\tau)$. Hence $R_2 \subseteq N_G(\Omega_1(V_3'))$. However $|\Omega_1(V_3')| = 3$ implies there is a subgroup $R_{2,1} \subseteq R_2$ such that $|R_2/R_{2,1}| \leqslant 2$ and $R_{2,1} \subseteq C_G(\Omega_1(V_3'))$. Hence, $R_{2,1} \subseteq N_G(G_3)$. However, $|R_2/R_{2,1}| \leqslant 2$ implies $|\bar{R}_2/\bar{R}_{2,1}| \leqslant 2$. Further, $|\bar{R}_2| \geqslant 8$ implies $|\bar{R}_{2,1}| \geqslant 4$. Since $\bar{R}_{2,1} \subseteq N_G(\bar{G}_3)$, $4 \mid |N_G(\bar{G}_3)|$ which is a contradiction.

Thus, every involution in $Z(V_2)$ acts fixed-point-free on V_3; which implies $Z(V_2)$ is cyclic. Let $Z(V_2) = \langle v \rangle$.

Since $R_2/Z(V_2)$ has 2 rank at least 3, there is an element $\tau_1 \in R_2 - Z(V_2)$ such that $\tau_1^2 = 1$. $R_2/Z(V_2)$ elementary abelian and $Z(V_2) \subseteq Z(R_2)$ imply R_2 has class at most 2. Let $x \in R_2$, then $1 = [\tau_1^2, x] = [\tau_1, x]^\tau [\tau_1, x]$. Since $[\tau, x] \in Z(V_2)$, we see $[\tau, x]^2 = 1$. Hence $\langle [\tau_1, x] \mid x \in R_2 \rangle \subseteq \Omega_1(Z(V_2))$. Thus $|R_2/C_{R_2}(\tau_1)| \leqslant 2$.

Since $R_2/Z(V_2)$ has rank at least 3 and $|R_2/C_{R_2}(\tau_1)| \leqslant 2$, there is an element $\alpha \in R_2 - Z(V_2) \times \langle \tau_1 \rangle$ such that $\alpha \in C_{R_2}(\tau_1)$. We will show $\langle \alpha, Z(V_2), \tau_1 \rangle$ contains U, an elementary abelian group of order 8.

Since $\alpha^2 \in V_2 \cap R_2$, $\alpha^2 \in Z(V_2)$. Thus $\alpha^2 = v^j$. If $2 \mid j$; then $\alpha^2 = v^{2j'}$ so that $(\alpha v^{-j'})^2 = 1$. If $\beta = \alpha v^{-j'}$, then $U = \langle \tau_1 \rangle \times \langle \beta \rangle \times \Omega_1(Z(V_2))$. Thus we may assume $\langle \alpha \rangle \supsetneq Z(V_2)$. However, $\langle \gamma \rangle \subseteq N_R(R_2)$, where γ acts transitively on the cosets of $Z(V_2)$ in R_2. Hence $\alpha^{\gamma^i} = \tau_1 v^k$, for some i and k. However $|\langle \tau_1 v^k \rangle| \leqslant |Z(V_2)|$ which contradicts $|\langle \alpha^{\gamma^i} \rangle| = |\langle \alpha \rangle| = 2\,|Z(V_2)|$. Hence U exists.

Since $U \subseteq N_G(V_3)$, applications of Theorem 5.3.16 [3] imply there is a subgroup $F \subseteq U$ such that $C_F(V_3) = V_3' = 1$ and $|U/F| = 2$. Since $\Omega_1(Z(V_2))$ acts fixed point free on $V_3^{\#}, F \cap Z(V_2) = 1$. Hence, $F \subseteq R_2$ implies $F \cap V_2 = 1$. Since $|U/F| \leqslant 2$; $|F| \geqslant 4$. Thus if $V_2 \neq 1$, the subgroup F still exists and the theorem is proved.

References

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